Conserved quantities in integrable systems

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Motivation

• 1D integrable models with strong interactions: s-1/2 Heisenberg, Hubbard, supersymmetric t-J

• local operators $Q_i : [Q_i, H] = [Q_i, Q_j] = 0$

Plan:

• How $Q_i$ influence transport properties & relaxation
• How to find all local and quasilocal $Q_i$
• How to describe nearly integrable systems
Generic systems relax towards thermal states

Necessary elements of thermalization (e.g., N. Linden, PRE, 2009):

- Equilibration: $\langle A(t) \rangle$ almost $t$-independent (steady state).
- Independence of the initial state (except its energy)
- Gibbs form: $\rho \propto \exp(-\beta H)$

But integrable system with $L$ sites:

- Has $L$ local conservation laws $[Q_l, H] = [Q_l, Q_{l'}] = 0$
- “Never forgets” initial values of $\langle Q_l(t) \rangle = \langle Q_l(0) \rangle$

For noninteracting particles $H = \sum_k E_k \hat{n}_k$, $\hat{n}_p$ are conserved
Some hallmarks of integrability:

**Ballistic transport:**

\[ F = \text{const} \quad F = 0 \]

\[ \left[ J, H \right] \neq 0 \]

\[ D = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \langle J(t) J \rangle \]

\[ D \geq \sum_l \frac{\langle J Q_l \rangle^2}{\langle Q_l^2 \rangle} \]
Signatures of charge stiffness beyond LR

\[ J = \sum_j i \ c_{j+1}^\dagger c_j + \text{h.c.}, \quad J' = \sum_j i c_{j+2}^\dagger (n_{j+1} - \frac{1}{2}) c_j + \text{h.c.} \]
Density of the entropy

Entropy $S_N(t) = -\text{Tr}_N \rho(t) \log \rho(t)$

generic

integrable
Generalized Gibbs Ensemble (GGE)

- Gibbs state – maximize entropy assuming $<H>$
  \[ \rho \propto \exp(-\beta H) \]

- GGE state – maximize entropy assuming $<H>$ and all other conserved quantities $<Q_i>$
  \[ \rho \propto \exp(-\lambda_1 H + \sum_{l \geq 2} \lambda_l Q_l) \]

- (Rigol 2007, see also concept by Jaynes 1957)
Do we know all $Q_i$?

\[ \rho \propto \exp(-\beta H - \sum_l \lambda_l Q_l) \quad \rightarrow \quad D = \sum_l \frac{\langle JQ_l \rangle^2}{\langle Q_l^2 \rangle} \]


Implications for the easy plane Heisenberg model:

\[ D > 0 \text{ while } \langle JQ_l \rangle = 0 \text{ for all known } Q_l \]

\[ \rho \neq \exp(-\beta H - \sum_l \lambda_l Q_l) \quad ? \quad \text{there exist other } Q_l \]
Why only local operators matter?

Local operators $O$ with support on $M$ sites

$$H|E_n\rangle = E_n|E_n\rangle$$

$$|E_n\rangle \langle E_n|$$

Local: $\sum_i \vec{s}_i \cdot \vec{s}_{i+1}$

Quasilocal: $\sum_i \vec{s}_i \cdot \vec{s}_{i+1} + \frac{1}{2} \vec{s}_i \cdot \vec{s}_{i+2} + \frac{1}{4} \vec{s}_i \cdot \vec{s}_{i+3} + \frac{1}{8} \vec{s}_i \cdot \vec{s}_{i+4} + \ldots$

Generic: $\sum_i \vec{s}_i \cdot \vec{s}_{i+1} + \vec{s}_i \cdot \vec{s}_{i+2} + \vec{s}_i \cdot \vec{s}_{i+3} + \vec{s}_i \cdot \vec{s}_{i+4} + \ldots$

$$D \geq \sum_l \frac{\langle JQ_l \rangle^2}{\langle Q_l^2 \rangle} = \sum_l \frac{\langle JQ_l^M \rangle^2}{\langle Q_l^2 \rangle}$$
General idea for constructing $Q_i$

1. Time averaged (conserved) $\bar{O} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \, O(t)$

2. Local $O$ usually lead to nonlocal $\bar{O}$

\[ \ldots 1 \otimes 1 \otimes 1 \otimes \neq 1 \otimes \ldots \]
\[ \ldots \otimes \neq 1 \]
\[ \otimes 1 \otimes 1 \otimes 1 \ldots + \text{translations} \]

3. Challenge: for $L \to \infty$ split $\bar{O} = O^M + O^\perp$:

(i) local $\|O^M\| > 0 \& \|O^\perp\| = 0$

(ii) quasilocal $\|O^M\| > 0 \& \|O^\perp\| > 0$

(iii) generic $\|O^M\| = 0$, e.g. $|E_n\rangle \langle E_n|$
The algorithm for generating $Q_i$

1. Generate all local operators $\langle O_\alpha O_\beta \rangle = \delta_{\alpha,\beta}$

2. Solve eigenproblem for matrix $K_{\alpha\beta} = \langle \bar{O}_\alpha \bar{O}_\beta \rangle$

3. Eigenvalue $\lambda$ and eigenvector $(v_1, v_2, v_3, v_4, \ldots)$

$$Q = \sum_s v_\alpha \bar{O}_\alpha$$

$$Q = Q^M + Q^\perp: \quad \|Q^M\|^2 = \lambda \|Q\|^2$$

Frame: $\lambda = 1$; Quasilocal: $0 < \lambda < 1$; Nonlocal: $\lambda = 0$

Heisenberg chain

$$H = \sum_{j=1}^{L} \left[ \frac{1}{2} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) + \Delta S_j^z S_{j+1}^z \right]$$

$$O_\alpha = \ldots 1 \otimes 1 \otimes 1 \otimes A_i = 1, \sqrt{2} S^\pm, 2 S^z \otimes 1 \otimes 1 \otimes 1 \ldots$$

$$\langle O_\alpha O_\beta \rangle = \delta_{\alpha, \beta}$$

$$T \to \infty$$

Even/odd parity: $$O_\alpha \pm PO_\alpha P, \quad P = \Pi_j (S_j^+ + S_j^-)$$

R/I sectors: $$O_\alpha + O_\alpha^{\dagger} \text{ and } i(O_\alpha - O_\alpha^{\dagger})$$
Test: break integrability by 2\textsuperscript{nd} nn interaction

Eigenvalues for $M = 6$

R - red circles, I - blue crosses

Parity even

Parity odd

$\Delta = 0.5, \Delta_2 = 0.5$

$1/L$
Integrable easy plane \((\Delta < 1)\)

Eigenvalues for \(M = 6\)

R - red circles, I - blue crosses

Parity even

Parity odd

\[1/L\]
Integrable isotropic & easy axis cases \((\Delta \geq 1)\)

Eigenvalues for \(M = 6\)

R - red circles, I - blue crosses

Parity even  \hspace{1cm}  Parity odd

1/L
Eigenvalues isotropic case - more details
Quasilocal $Q'$ in even sector isotropic Heisenberg

\[
Q' = \sum_j \left[ \alpha S_{j+2} S_j + \zeta (S_{j+3} S_j) + \beta (S_{j+3} S_{j+2}) (S_{j+1} S_j) + \gamma (S_{j+3} S_{j+1}) (S_{j+2} S_j) \right] + \ldots + \ldots + \ldots
\]

\[
\alpha \simeq \beta \simeq \zeta \quad \gamma \simeq -2\alpha
\]
Number of $Q_i$

$0 \leq \lambda_i \leq 1 \rightarrow N_{Q_i} \geq \sum_i \lambda_i$

$\Delta = 0.5$

$\Delta = 1$

$\Delta = 1.5$

$\Delta = 0.5$, Even

$\Delta = 0.5$, Odd

Nonint

Bethe ansatz
Main results for integrable systems

• Algorithm for generating local and quasilocal conserved quantities or numerical testing of the integrability

• In Heisenbeg model, quasilocal conserved quantities (may) exist in all symmetry sectors

• Number of (local & quasilocal) conserved quantities is proportional to $M \rightarrow \text{GGE}$ (at least for weak quenches)
Some hallmarks of almost integrable systems:

Almost ballistic transport:

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \langle J(t) J \rangle = \sum_l \frac{\langle J Q_l \rangle^2}{\langle Q_l^2 \rangle}
\]

\[
\frac{1}{\tau} \int_0^\tau dt \langle J(t) J \rangle = ??
\]
Breaking of integrability: general expectations

\[ H = J \sum_{r=1}^{L} (S_r^x S_{r+1}^x + S_r^y S_{r+1}^y + \Delta S_r^z S_{r+1}^z + \Delta_2 S_r^z S_{r+2}^z) \]

If \( \Delta_2 \) is not ”too large” and time \( t \) not ”too long”:

- \( < A(t)A > \) stay large

- conserved quantities, \( Q_l \), depend on \( \Delta_2 \) (may fade out)

- local \( Q_l \) change into generic nonlocal \( |E_n\rangle\langle E_n| \)

- Is the change gradual or abrupt?
  Gradual change naturally involves quasilocal operators
Our main results on integrable systems (M-fixed)

\[ \langle Q_l Q_{l'} \rangle \propto \delta_{ll'} , \quad \text{independence} \]

\[
\langle \bar{A} A \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \langle A(t) A \rangle \\
= \sum_l \frac{\langle A Q_l \rangle^2}{\langle Q_l^2 \rangle} , \quad \text{Mazur} \\
= \sum_l \lambda_l \frac{\langle A Q_l^M \rangle^2}{\langle (Q_l^M)^2 \rangle} , \quad \text{locality}
\]
Finite time averaging for nearly integrable systems

\[ \langle Q_l Q_{l'} \rangle \propto \delta_{ll'}, \quad \text{independence} \]

\[ \langle \bar{A} A \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \langle A(t) A \rangle \]

\[ = \sum_l \frac{\langle AQ_l \rangle^2}{\langle Q_l^2 \rangle}, \quad \text{Mazur} \]

\[ = \sum_l \lambda_l \frac{\langle AQ^M_l \rangle^2}{\langle (Q^M_l)^2 \rangle}, \quad \text{locality} \]
Appropriate time-averaging

\[ \bar{A}^\tau = \sum_{m,n} \theta \left( \frac{1}{\tau} - |E_m - E_n| \right) \langle m|A|n \rangle \langle m|n \rangle \]

\[ \bar{A}^\tau = \frac{1}{\tau} \int_0^\tau dt A(t) \quad \text{only for } \tau \to \infty \text{ and } \tau \to 0 \]

How exotic is such time averaging?

\[ \langle \bar{A}^\tau B \rangle = \int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} d\omega \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle A(t) B \rangle \right] \]

broadening of \( \delta(\omega) \)
\[ \langle \tilde{A}^\tau A \rangle = \sum_{m,n} \theta \left( \frac{1}{\tau} - |E_m - E_n| \right) \left| \langle m | A | n \rangle \right|^2 \]
Eigenvalues (=support)

\[ \Delta = 0.5, M = 5 \]

\[
\langle \bar{A}^{\tau} A \rangle = \sum_l \lambda_l \frac{\langle A Q_l^M \rangle^2}{\langle (Q_l^M)^2 \rangle}
\]
Eigenoperators

\[ \Delta = 0.5, \; M = 5 \]

\[ \langle \tilde{A}^\tau A \rangle = \sum_l \lambda_l \frac{\langle AQ_l^M \rangle^2}{\langle Q_l^M \rangle^2} \]

\[ \cos(\alpha_l) = \frac{\langle Q_l^M Q_{l0}^M \rangle}{||Q_l^M|| \cdot ||Q_{l0}^M||} \]

\[ \Delta_2 \neq 0 \; \Delta_2 = 0 \]

\[ \text{ER sector} \quad \Delta_2 = \Delta/6 \]

\[ \text{O1 sector} \quad \Delta_2 = \Delta/6 \]
Dependence on $\Delta_2$

$\Delta = 0.5, M = 5$

\[
R_l(\Delta_2, \tau) = \frac{\lambda_l(L \to \infty, \tau, \Delta_2)}{\lambda_l(L \to \infty, \tau \to \infty, 0)} \approx \frac{2}{\pi} \arctan \left( \frac{1}{\Gamma_l \tau (\Delta_2)^2} \right)
\]

![Graphs showing dependence of $R_2$ and $R_3$ on $1/\tau \Delta_2^2$ for different values of $\Delta_2$.](image)
Dependence on $\Delta_2$

$\Delta = 0.5, M = 5$

\[ R_l(\Delta_2, \tau) = \frac{\lambda_l(L \to \infty, \tau, \Delta_2)}{\lambda_l(L \to \infty, \tau \to \infty, 0)} \approx \frac{2}{\pi} \arctan \left( \frac{1}{\Gamma_l \tau (\Delta_2)^2} \right) \]

Lowest order perturbation should hold for weak $\Delta_2$

**El sector**
- $\Delta_2 = \Delta/8$
- $\Delta_2 = \Delta/6$
- $\Delta_2 = \Delta/4$
- $2/\pi \cdot \text{atan}(x/0.14)$

**Ol sector**
- $\Delta_2 = \Delta/8$
- $\Delta_2 = \Delta/6$
- $\Delta_2 = \Delta/4$
- $2/\pi \cdot \text{atan}(x/0.43)$

thermal transport

spin transport
Test for $L=30$ (MCLM)

\[ \lambda_1 \propto \langle \bar{J}\tau J \rangle = \int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} d\omega \frac{1}{2\pi} \int_{-\infty}^{\infty} dte^{i\omega t} \langle J(t)J \rangle \]

Rescaled data for spin current

Raw data for energy current
Main results for nearly integrable systems

\[
\frac{1}{\tau} \int_0^\tau dt \langle A(t)B \rangle \sim \langle \tilde{A}^\tau B \rangle = \sum \lambda_l \frac{\langle AQ_l^M \rangle^2}{\langle (Q_l^M)^2 \rangle}
\]

After breaking integrability by $\Delta_2$

- smooth changing of $Q_l(\tau, \Delta_2)$
- all local $Q_l$ become quasilocal (except for $H$)
- support of $Q_l$ expands perturbatively as $\lambda_l \propto \arctan \left( \frac{1}{\Gamma_l \tau (\Delta_2)^2} \right)$
- $\Gamma_l$ increase with support, but are of the same order of magnitude (for small $M$)