Conserved quantities in integrable systems

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Motivation

- 1D integrable models with strong interactions:

  s-1/2 Heisenberg, Hubbard, supersymmetric t-*J*
- local operators $Q_i \quad : \quad [Q_i, H] = [Q_i, Q_j] = 0$

Plan:

- How Q_i influence transport properties & relaxation
- How to find all local and quasilocal Q_i
- How to describe nearly integrable systems

Generic systems relax towards thermal states

Necessary elements of thermalization

(e.g, N. Linden, PRE, 2009):

- Equilibration: <*A(t)*> almost *t*-independent (steady state).
- Independence of the initial state (except its energy)
- Gibbs form: $\rho \propto \exp(-\beta H)$

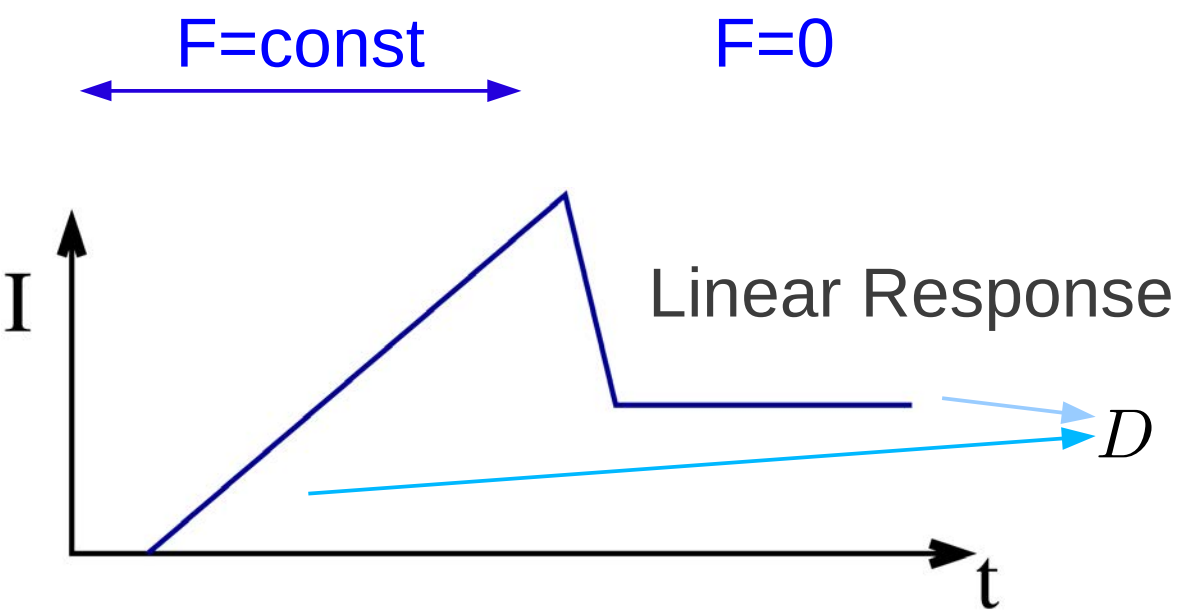
But integrable system with *L sites*:

- Has *L* local conservation laws $[Q_l, H] = [Q_l, Q_{l'}] = 0$
- "Never forgets" initial values of $\langle Q_l(t) \rangle = \langle Q_l(0) \rangle$

For noninteracting particles $H = \sum_k E_k \hat{n}_k$, $\hat{n}_p$ are conserved

Some hallmarks of integrability:

Ballistic transport:

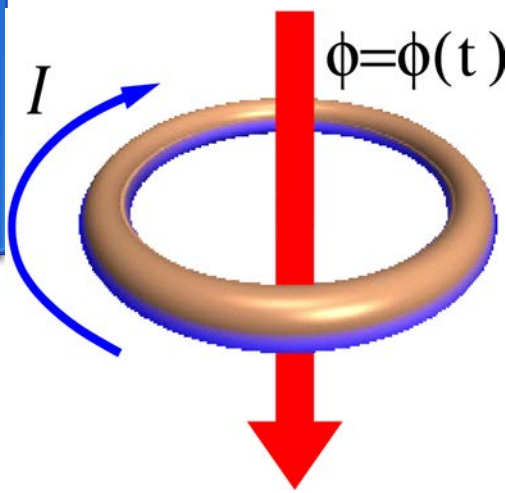


$$[\mathrm{J},\mathrm{H}] \neq 0$$

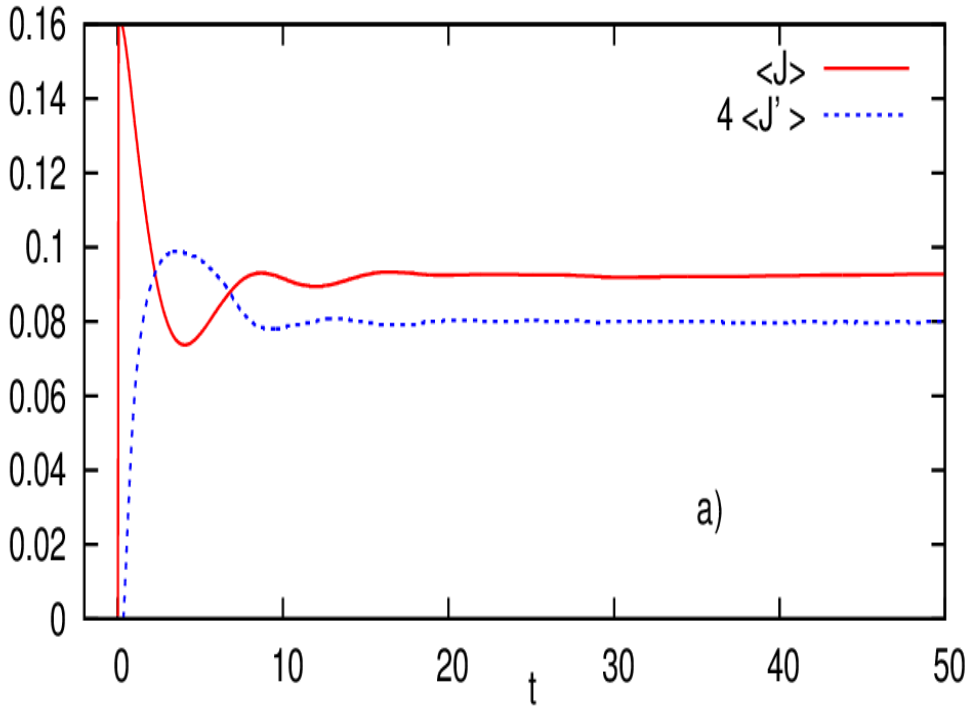
$$D = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau dt \langle J(t) J \rangle$$

$$D \geq \sum_l \frac{\langle J Q_l \rangle^2}{\langle Q_l^2 \rangle}$$

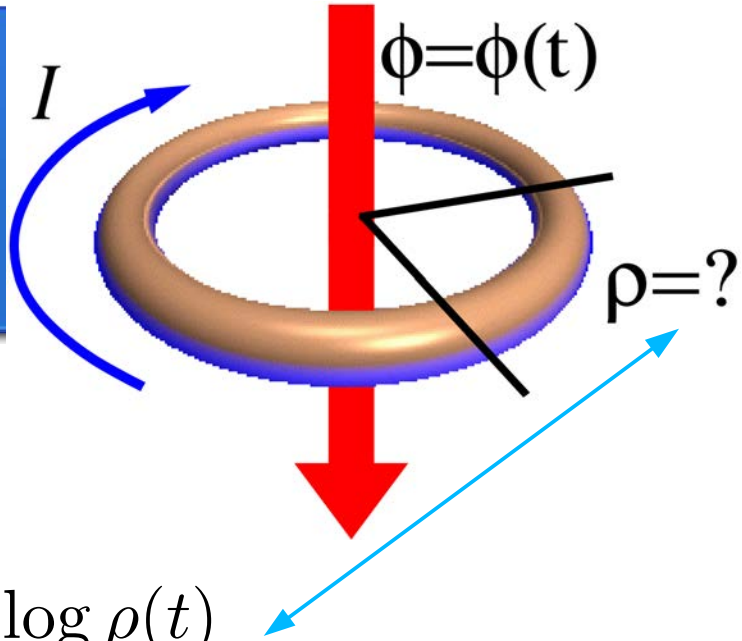
Signatures of charge stiffness beyond LR



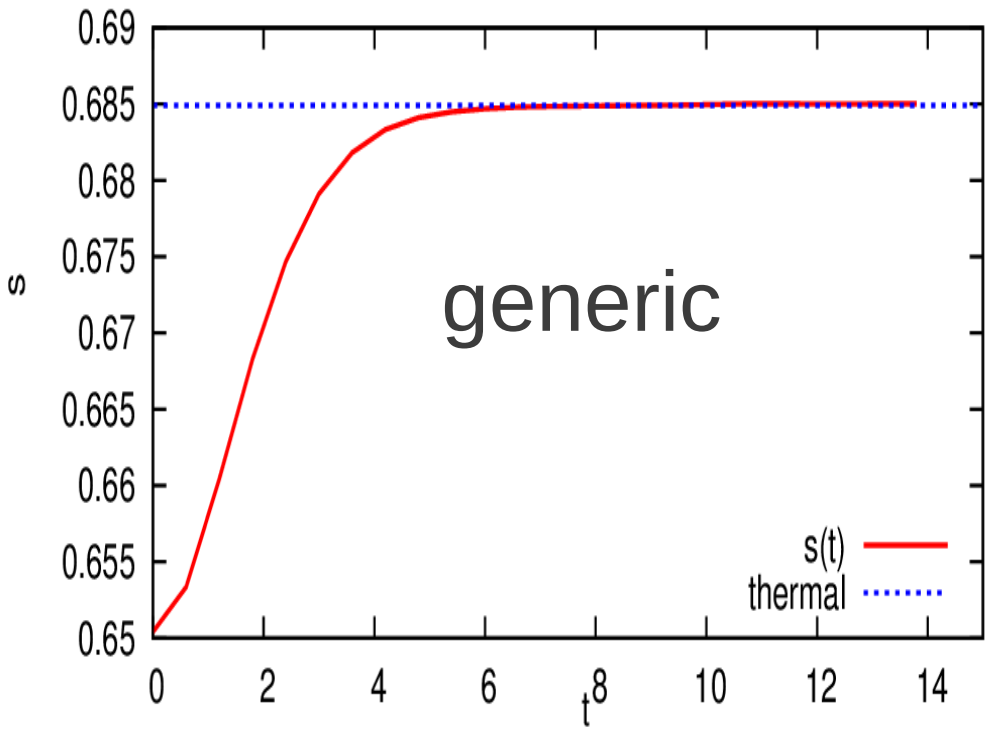
$$J = \sum_j i\, c^\dagger_{j+1} c_j + \text{h.c.}, \qquad J' = \sum_j i c^\dagger_{j+2} (n_{j+1} - \tfrac{1}{2}) c_j + \text{h.c.}$$

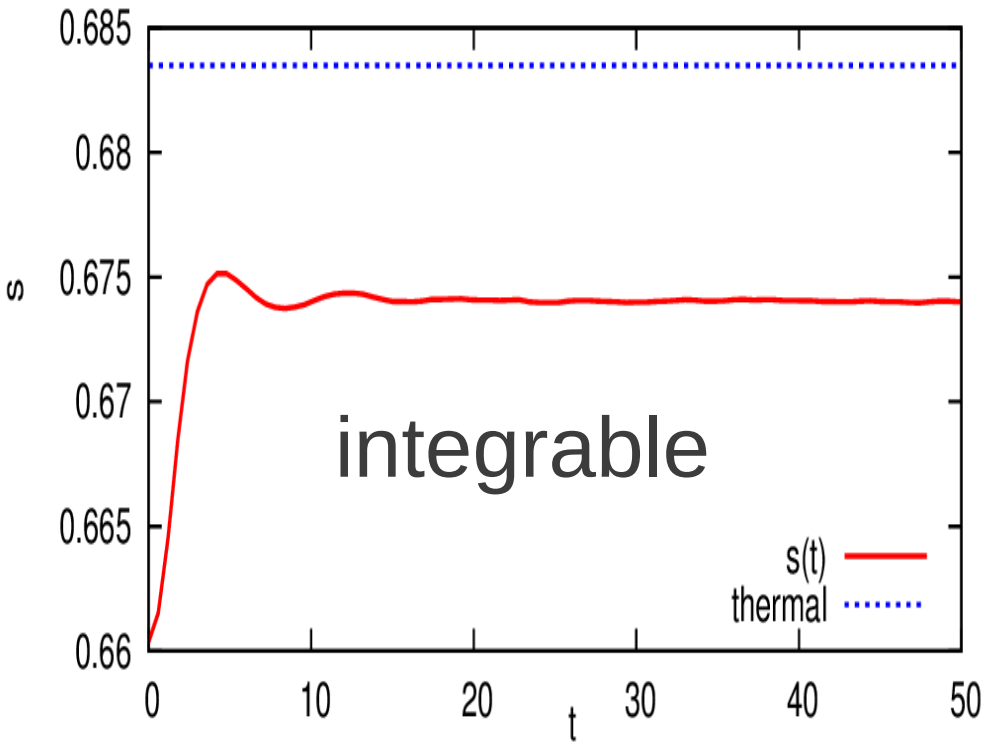


Density of the entropy



$$\text{Entropy } S_N(t) = -\mathrm{Tr}_N \rho(t) \log \rho(t)$$





Generalized Gibbs Ensemble (GGE)

- Gibbs state – maximize entropy assuming <*H*>

$$\rho \propto \exp(-\beta H)$$

- GGE state – maximize entropy assuming <*H*> and all other conserved quantities <Q_i>

$$\rho \propto \exp(-\lambda_1 H + \sum_{l \geq 2} \lambda_l Q_l)$$

- (Rigol 2007, see also concept by Jaynes 1957)

Do we know all Q_i ?

$$\rho \propto \exp(-\beta H - \textstyle\sum_l \lambda_l Q_l) \Longrightarrow D = \sum_l \frac{\langle J Q_l \rangle^2}{\langle Q_l^2 \rangle}$$

M.M. P. Prelovsek. T. Prosen, PRL (2014)

Implications for the easy plane Heisenberg model:

$$D > 0 \text{ while } \langle J Q_l \rangle = 0 \text{ for all known } Q_l$$

?

$$\rho \neq \exp(-\beta H - \textstyle\sum_l \lambda_l Q_l)$$

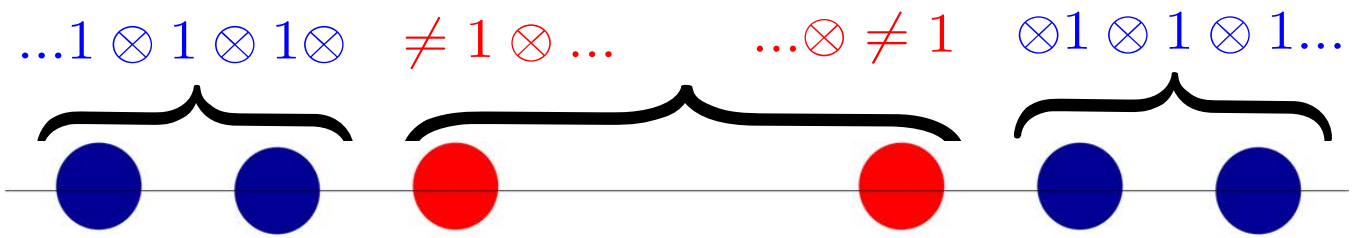
there exist other Q_l

Why only local operators matter ?

$H|E_n\rangle = E_n|E_n\rangle$

$|E_n\rangle\langle E_n|$

Local operators O with support on M sites



local : $\sum_i \vec{s}_i \cdot \vec{s}_{i+1}$

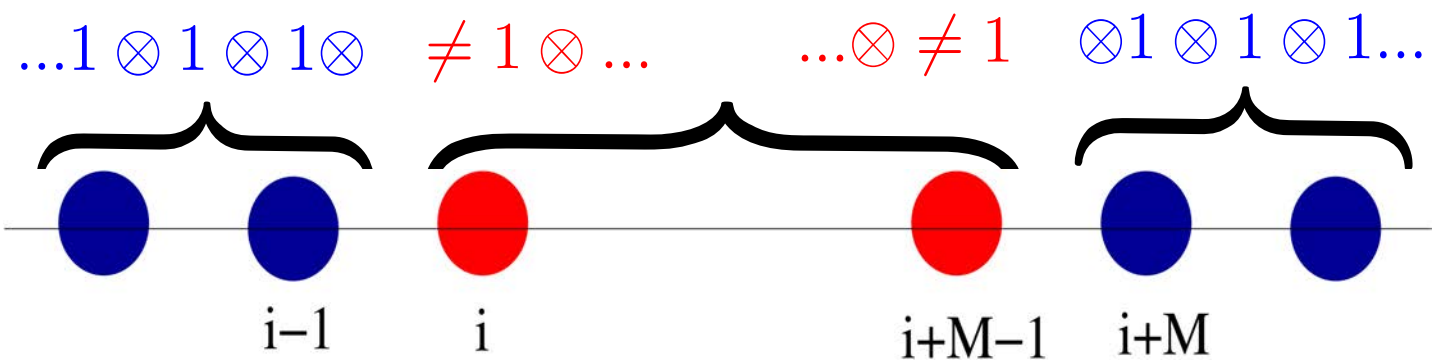
quasilocal: $\sum_i \vec{s}_i \cdot \vec{s}_{i+1} + \frac{1}{2}\vec{s}_i \cdot \vec{s}_{i+2} + \frac{1}{4}\vec{s}_i \cdot \vec{s}_{i+3} + \frac{1}{8}\vec{s}_i \cdot \vec{s}_{i+4} + \ldots.$

generic : $\sum_i \vec{s}_i \cdot \vec{s}_{i+1} + \vec{s}_i \cdot \vec{s}_{i+2} + \vec{s}_i \cdot \vec{s}_{i+3} + \vec{s}_i \cdot \vec{s}_{i+4} + \ldots.$

$$D \geq \sum_l \frac{\langle JQ_l\rangle^2}{\langle Q_l^2\rangle} = \sum_l \frac{\langle JQ_l^M\rangle^2}{\langle Q_l^2\rangle}$$

General idea for constructing Q_i

1. Time averaged (conserved) $\bar{O} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau dt\, O(t)$
2. Local O usually lead to nonlocal $\bar{O}$



+translations

3. Challenge: for $L \to \infty$ split $\bar{O} = O^M + O^\perp$:

(i) local $\|O^M\| > 0$ & $\|O^\perp\| = 0$
(ii) quasilocal $\|O^M\| > 0$ & $\|O^\perp\| > 0$
(iii) generic $\|O^M\| = 0$, e.g. $|E_n\rangle\langle E_n|$

The algorithm for generating Q_i

1. Generate all local operators $\langle O_\alpha O_\beta \rangle = \delta_{\alpha,\beta}$

2. Solve eigenproblem for matrix $K_{\alpha\beta} = \langle \bar{O}_\alpha \bar{O}_\beta \rangle$

3. Eigenvalue λ and eigenvector $(v_1, v_2, v_3, v_4, ...)$

$$\mathrm{Q} = \sum_s v_\alpha \bar{O}_\alpha$$

$$Q = Q^M + Q^\perp: \qquad \|Q^M\|^2 = \lambda \|Q\|^2$$

Local: $\lambda = 1$; Quasilocal: $0 < \lambda < 1$; Nonlocal: $\lambda = 0$

M. M. P. Prelovsek. T. Prosen, PRL (2015)

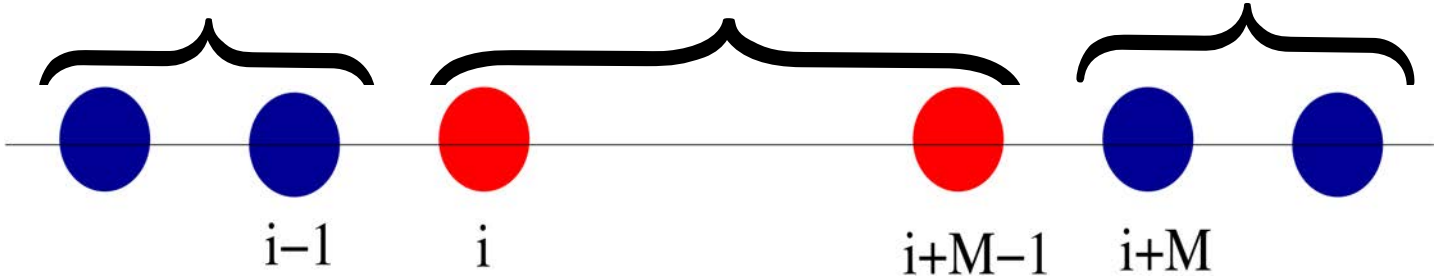
Heisenberg chain

$$H = \sum_{j=1}^{L} \left[\frac{1}{2}(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) + \Delta S_j^z S_{j+1}^z \right]$$

$$O_\alpha = \ldots 1 \otimes 1 \otimes 1 \otimes \quad A_i = 1, \sqrt{2}S^\pm, 2S^z \quad \otimes 1 \otimes 1 \otimes 1 \ldots$$

$$\langle O_\alpha O_\beta \rangle = \delta_{\alpha,\beta}$$

$$\mathrm{T} \to \infty$$



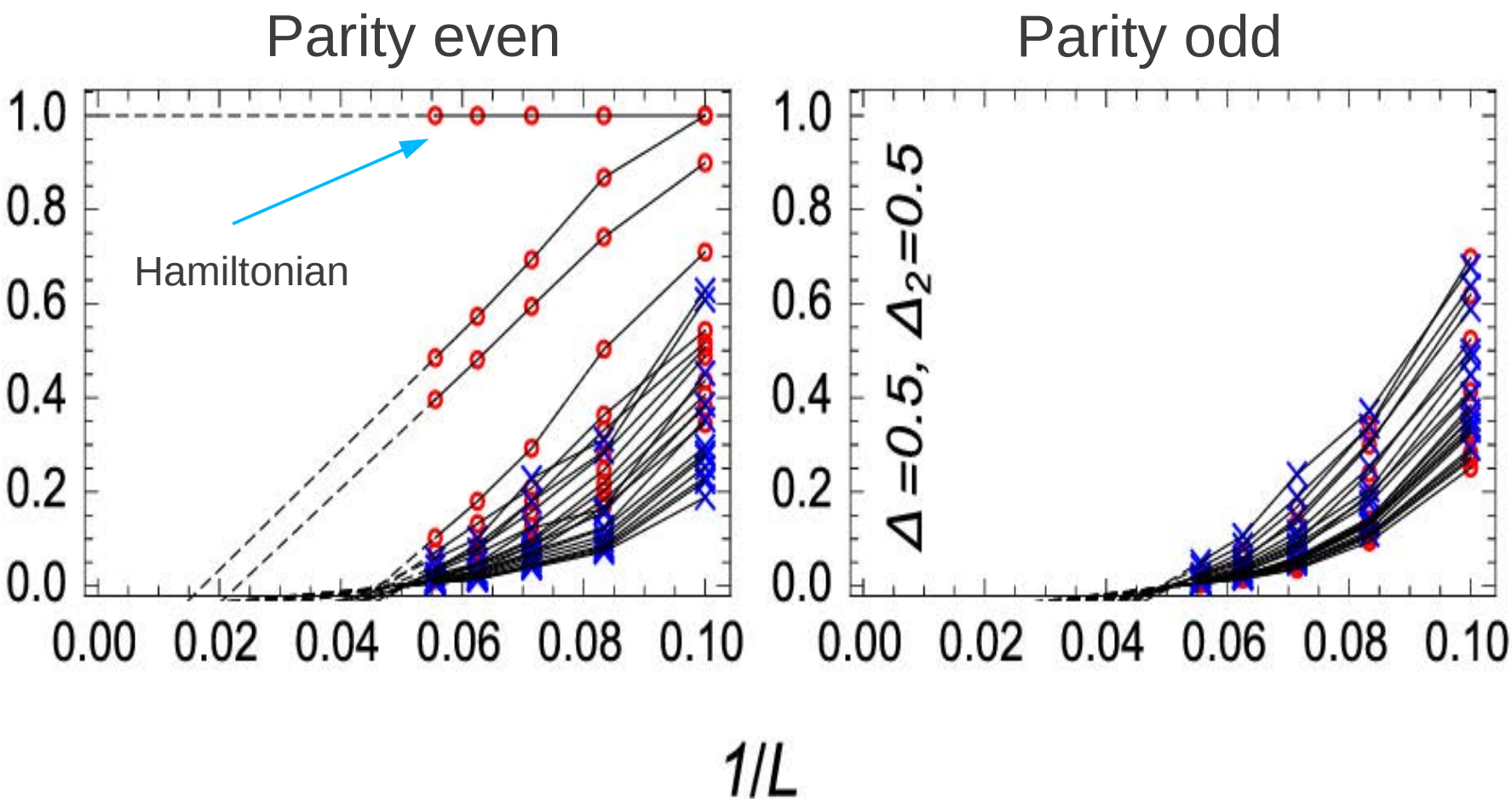
Even/odd parity : $O_\alpha \pm P O_\alpha P$, $\qquad P = \Pi_j (S_j^+ + S_j^-)$

R/I sectors: $O_\alpha + O_\alpha^\dagger$ and $i(O_\alpha - O_\alpha^\dagger)$

Test: break integrability by 2nd nn interaction

Eigenvalues for $M = 6$

R - red circles, I -blue crosses



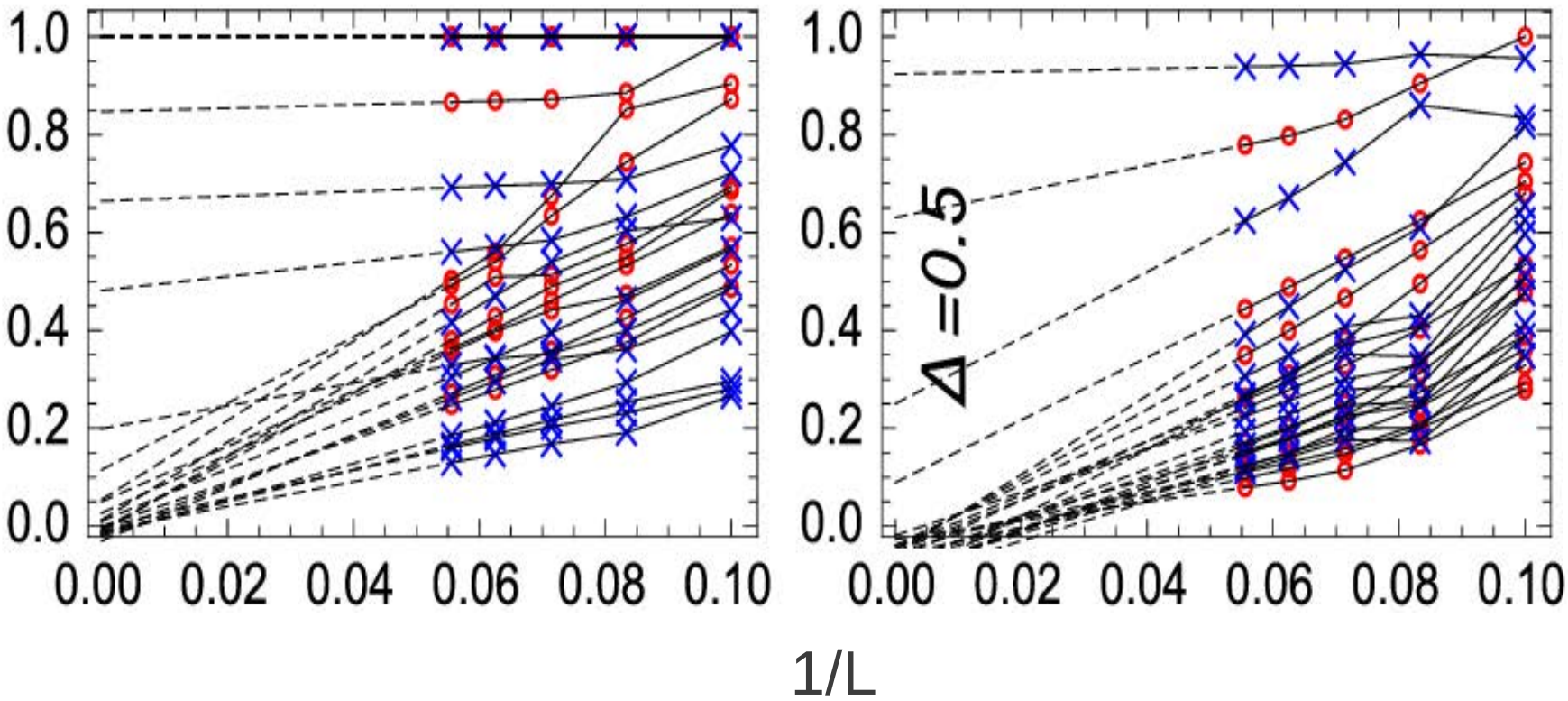
Integrable easy plane $(\Delta < 1)$

Eigenvalues for $M = 6$

R - red circles, I -blue crosses

Parity even

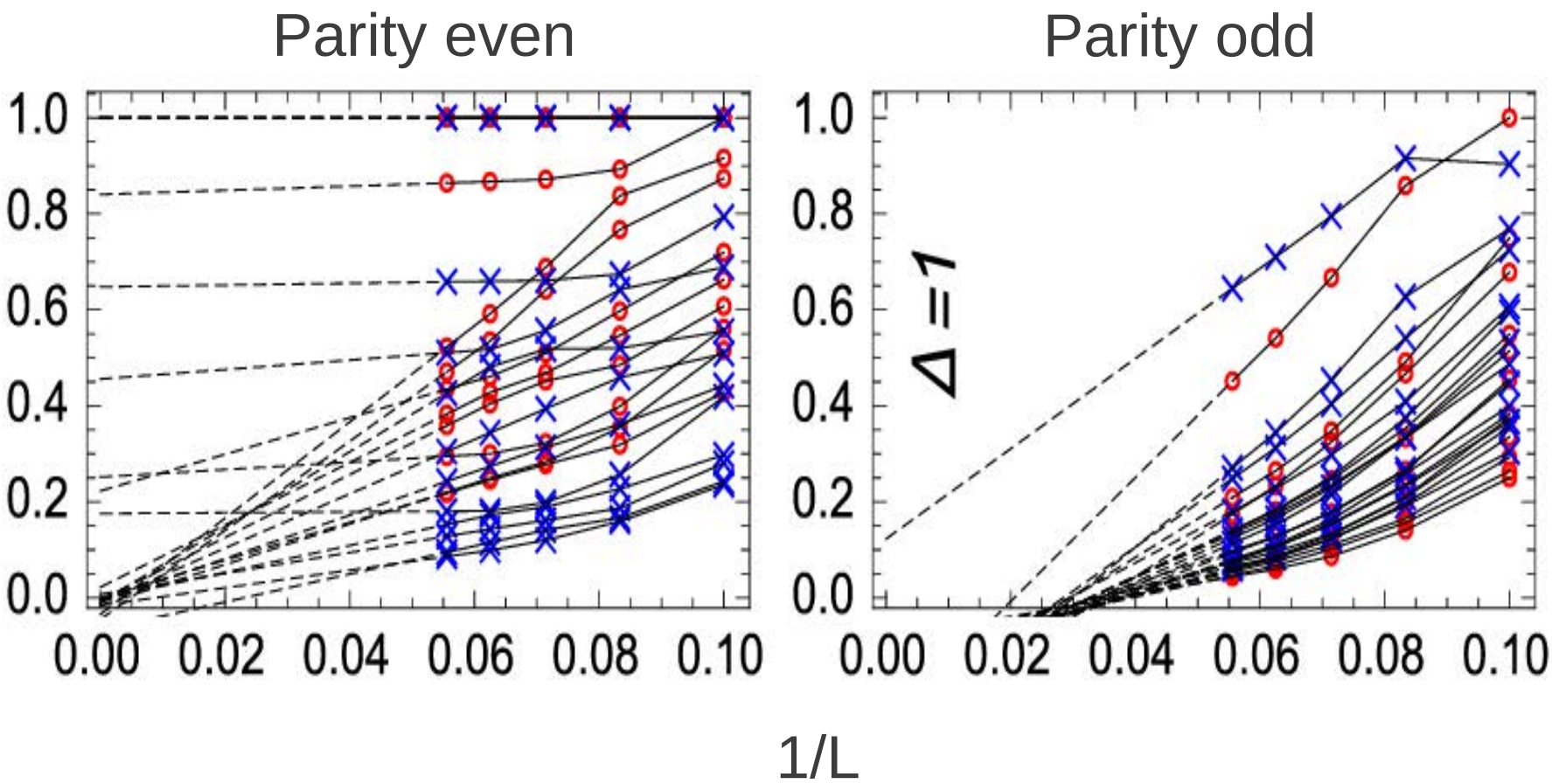
Parity odd



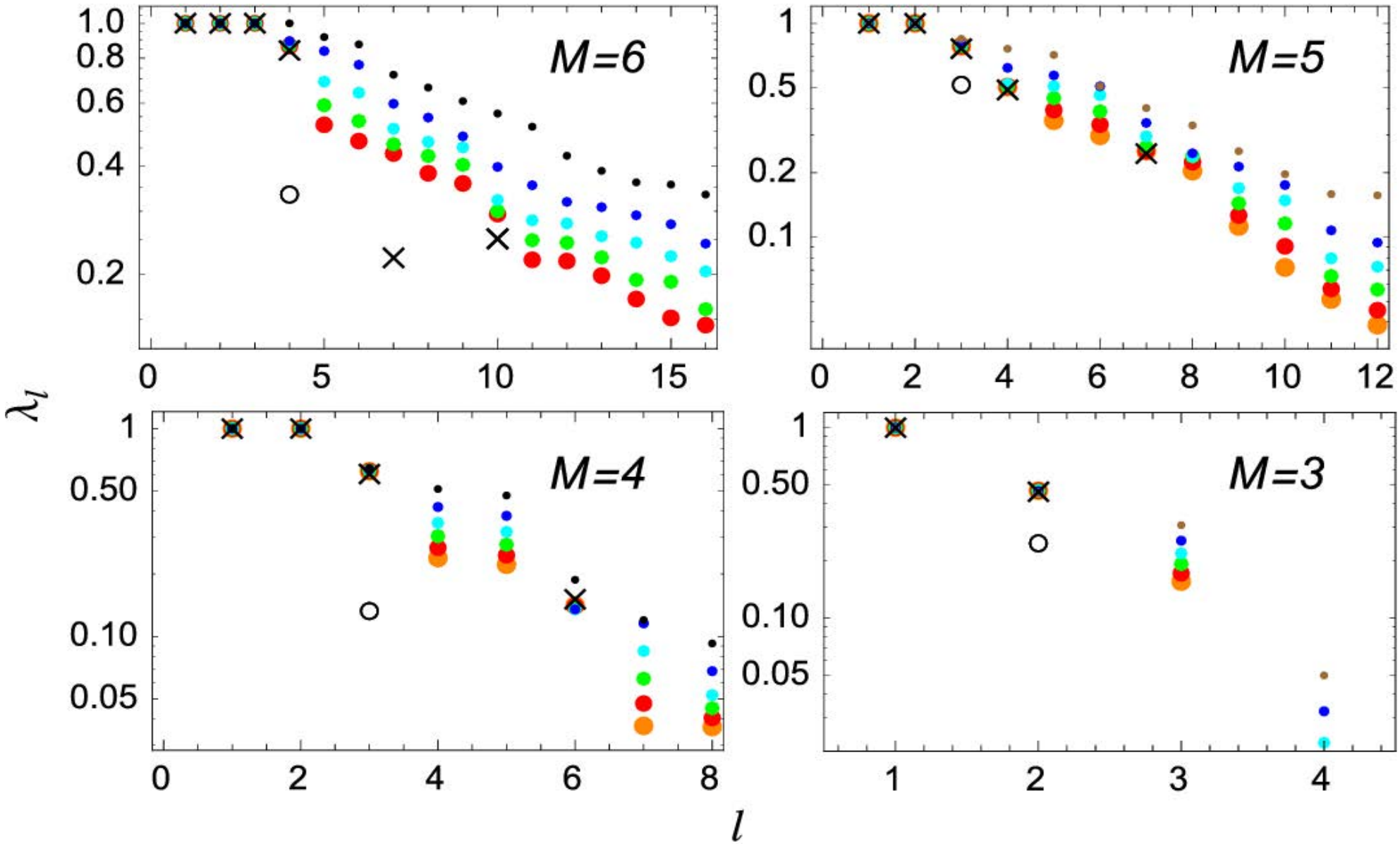
Integrable isotropic & easy axis cases $(\Delta \geq 1)$

Eigenvalues for $M = 6$

R - red circles, I -blue crosses



Eigenvalues isotropic case -more details



Quasilocal Q in even sector isotropic Heisenberg

$$Q' =$$

$$\sum_j [\alpha \mathbf{S}_{j+2}\mathbf{S}_j + \zeta(\mathbf{S}_{j+3}\mathbf{S}_j)$$

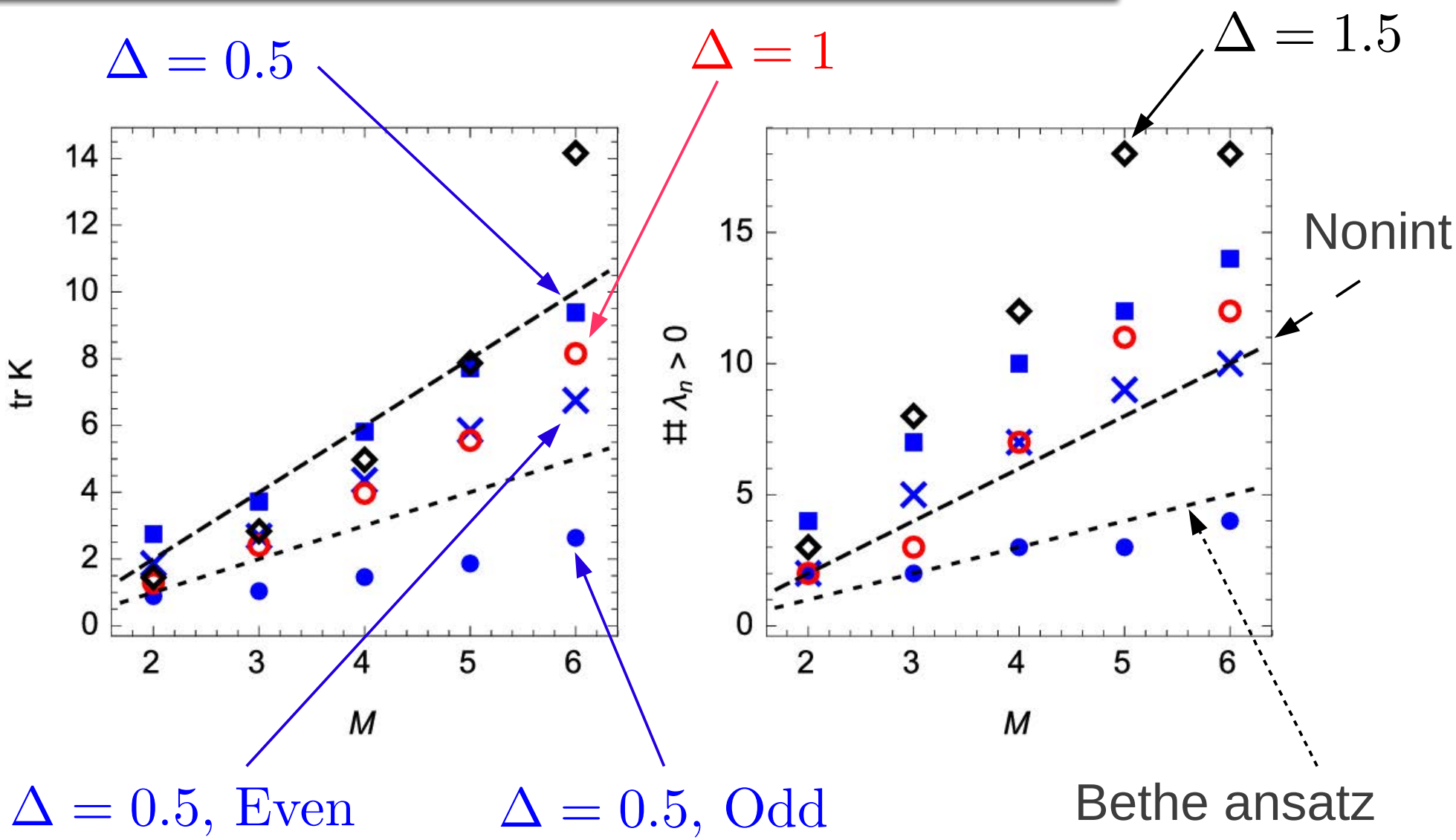
$$+\beta(\mathbf{S}_{j+3}\mathbf{S}_{j+2})(\mathbf{S}_{j+1}\mathbf{S}_j) + \gamma(\mathbf{S}_{j+3}\mathbf{S}_{j+1})(\mathbf{S}_{j+2}\mathbf{S}_j)]$$

$$+ \ldots\ldots + \ldots\ldots\ldots + ..$$

$$\alpha \simeq \beta \simeq \zeta \qquad\qquad \gamma \simeq -2\alpha$$

Number of Q_i

$$0 \leq \lambda_i \leq 1 \quad \rightarrow \quad N_{Qi} \geq \sum_i \lambda_i$$

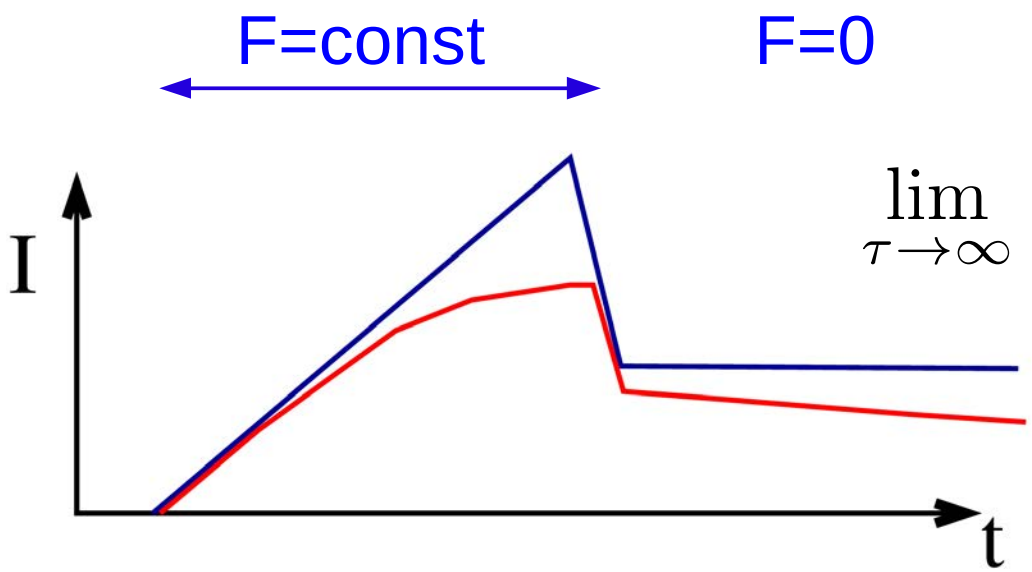


Main results for integrable systems

- Algorithm for generating local and quasilocal conserved quantities or numerical testing of the integrability
- In Heisenbeg model, quasilocal conserved quantities (may) exist in all symmetry sectors
- Number of (local & quasilocal) conserved quantities is proportional to M → GGE (at least for weak quenches)

Some hallmarks of almost integrable systems:

Almost ballistic transport:



$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau dt\langle J(t)J\rangle = \sum_l \frac{\langle JQ_l\rangle^2}{\langle Q_l^2\rangle}$$

$$\frac{1}{\tau}\int_0^\tau dt\langle J(t)J\rangle = ??$$

Breaking of integrability: general expectations

$$H = J \sum_{r=1}^{L} (S_r^x S_{r+1}^x + S_r^y S_{r+1}^y + \Delta S_r^z S_{r+1}^z + \Delta_2 S_r^z S_{r+2}^z)$$

If Δ_2 is not "too large" and time t not "too long":

- $< A(t)A >$ stay large
- conserved quantities, Q_l, depend on Δ_2 (may fade out)
- local Q_l change into generic nonlocal $|E_n\rangle\langle E_n|$

$\Updownarrow$

- Is the change gradual or abrupt ?
  Gradual change naturally involves quasilocal operators

Our main results on integrable systems (M-fixed)

$$\langle Q_l Q_{l'} \rangle \propto \delta_{ll'}, \qquad \text{independence}$$

$$\begin{aligned}
\langle \bar{A} A \rangle &= \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt \langle A(t) A \rangle \\
&= \sum_l \frac{\langle A Q_l \rangle^2}{\langle Q_l^2 \rangle}, \qquad \text{Mazur} \\
&= \sum_l \lambda_l \frac{\langle A Q_l^M \rangle^2}{\langle (Q_l^M)^2 \rangle}, \qquad \text{locality}
\end{aligned}$$

Finite time averaging for nearly integrable systems

$$\langle Q_l Q_{l'} \rangle \propto \delta_{ll'}, \qquad \text{independence}$$

$$\begin{aligned}
\langle \bar{A} A \rangle &= \cancel{\lim_{\tau \to \infty}} \frac{1}{\tau} \int_0^\tau dt \langle A(t) A \rangle \\
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&= \sum_l \lambda_l \frac{\langle A Q_l^M \rangle^2}{\langle (Q_l^M)^2 \rangle}, \qquad \text{locality}
\end{aligned}$$

Appropriate time-averaging

$$\bar{A}^\tau = \sum_{m,n} \theta\left(\frac{1}{\tau} - |E_m - E_n|\right) \langle m|A|n\rangle \;\; |m\rangle\langle n|$$

$$\bar{A}^\tau = \frac{1}{\tau}\int_0^\tau dt A(t) \quad \text{only for } \tau \to \infty \text{ and } \tau \to 0$$

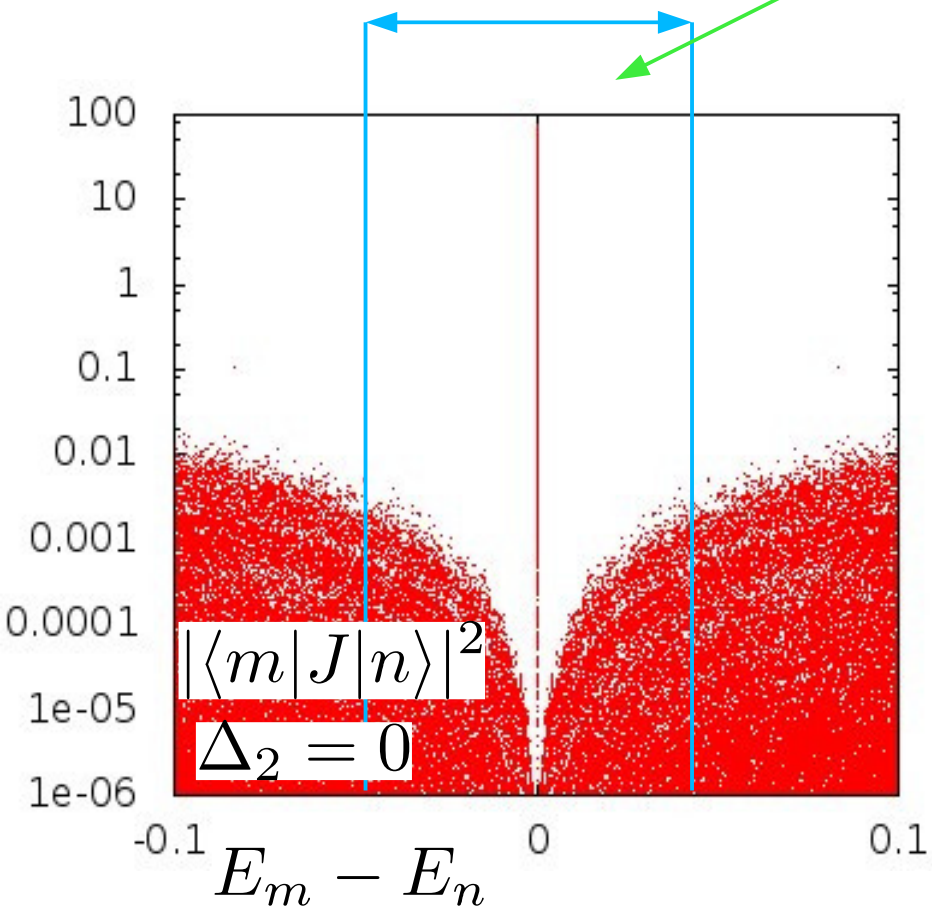
How exotic is such time averaging?

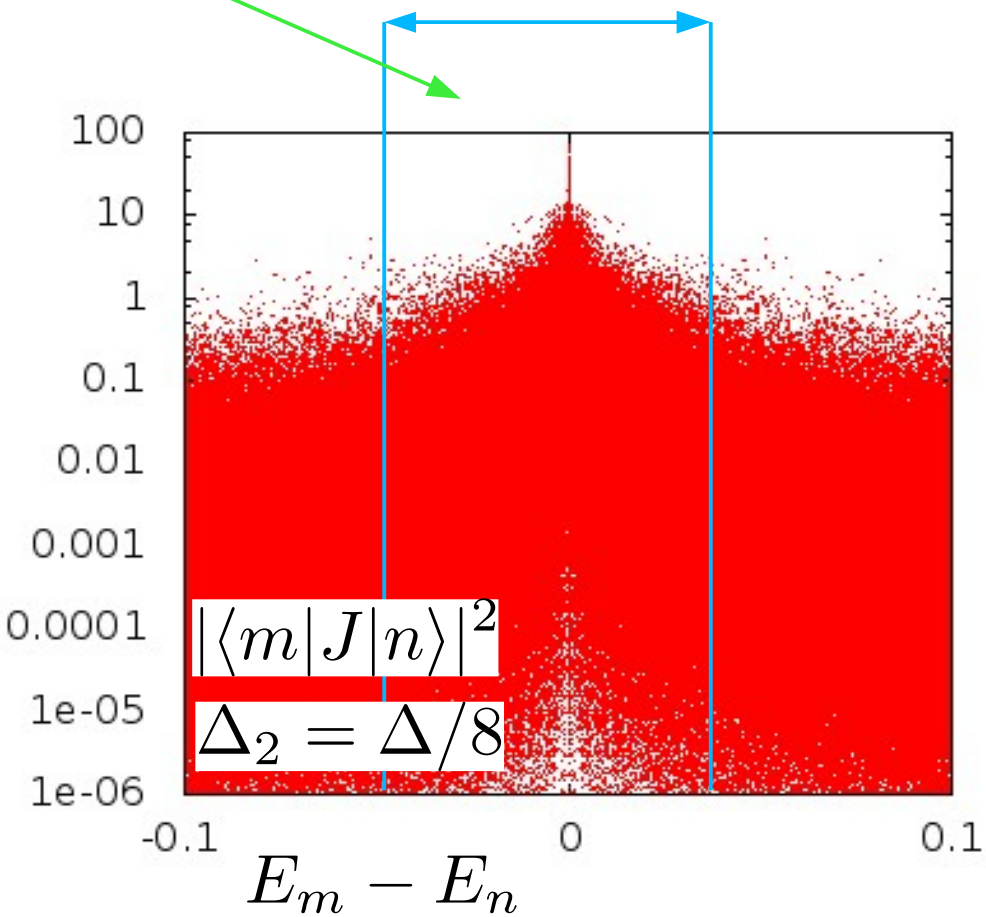
$$\langle \bar{A}^\tau B\rangle = \int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} d\omega \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dt e^{i\omega t}\langle A(t)B\rangle\right]$$

broadening of $\delta(\omega)$

Other perspective

$$\langle \bar{A}^\tau A \rangle = \sum_{m,n} \theta\left(\frac{1}{\tau} - |E_m - E_n|\right) |\langle m|A|n\rangle|^2$$

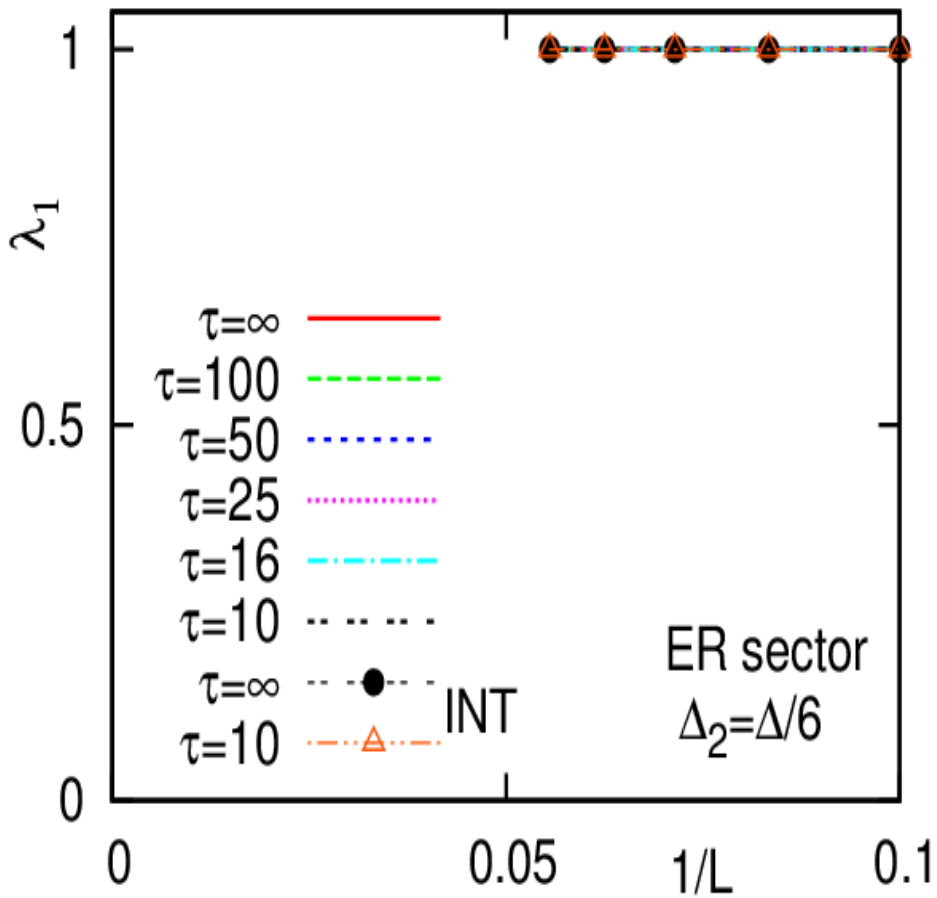


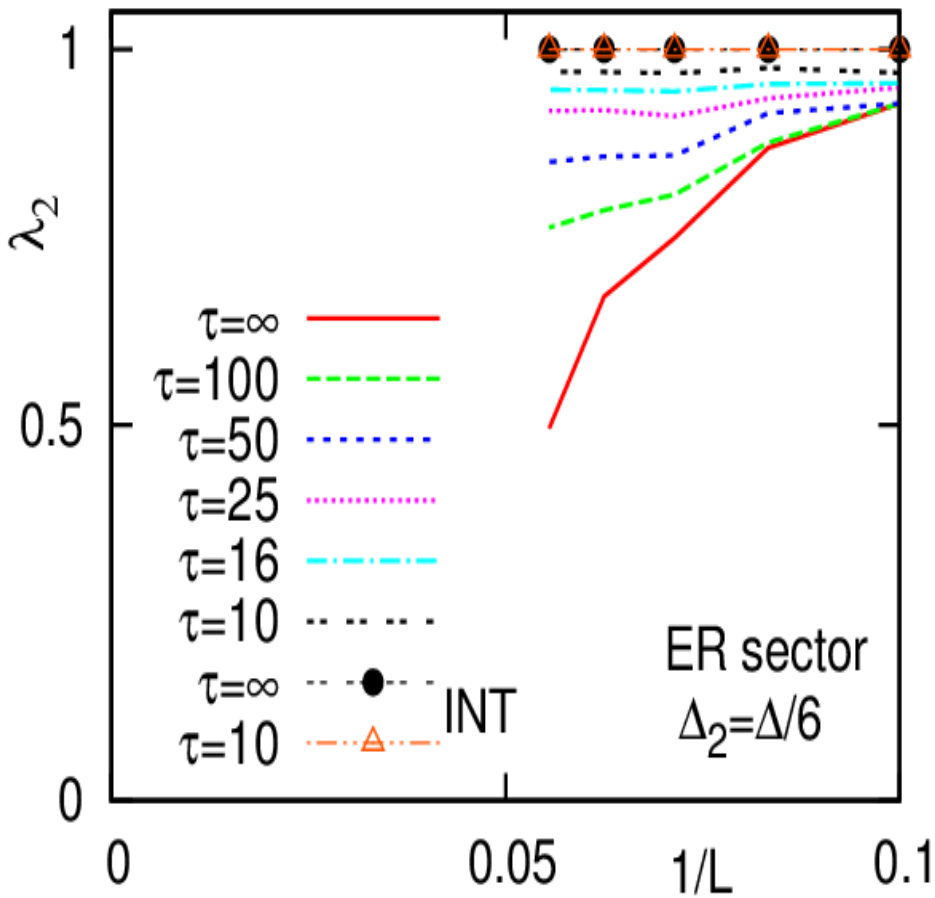


Eigenvalues (=support)

$\Delta = 0.5, M = 5$

$$\langle \bar{A}^\tau A \rangle = \sum_l \boxed{\lambda_l} \frac{\langle A Q_l^M \rangle^2}{\langle (Q_l^M)^2 \rangle}$$





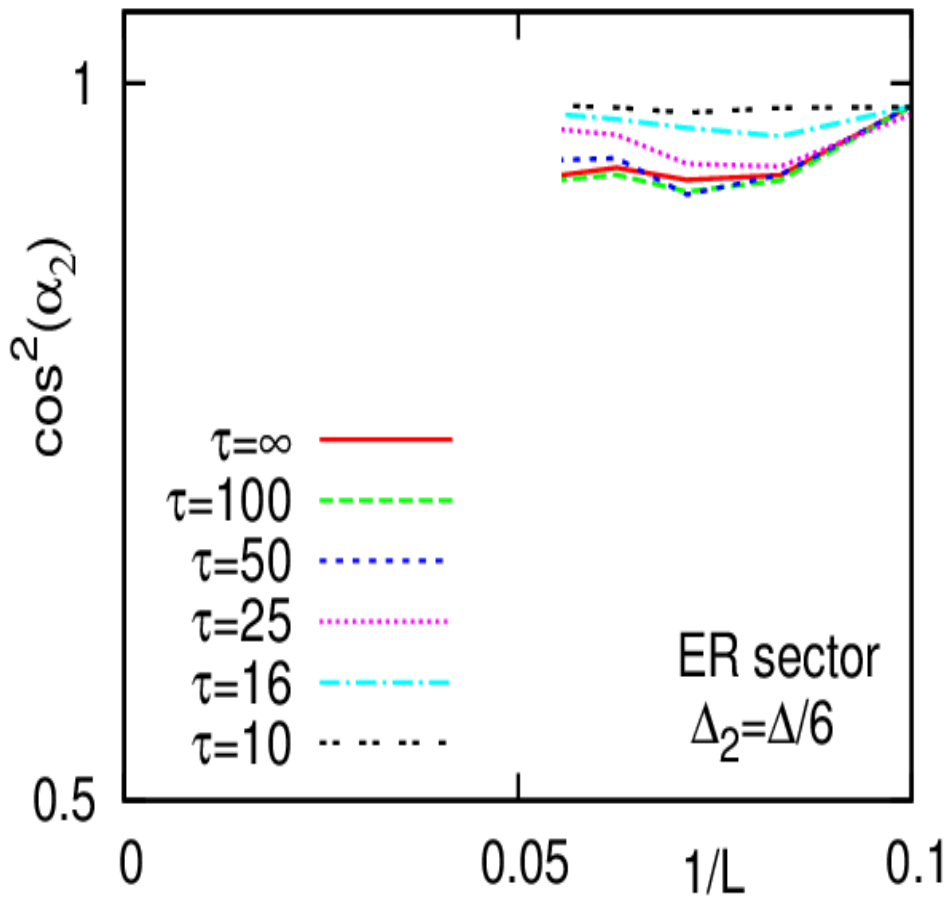
Eigenoperators

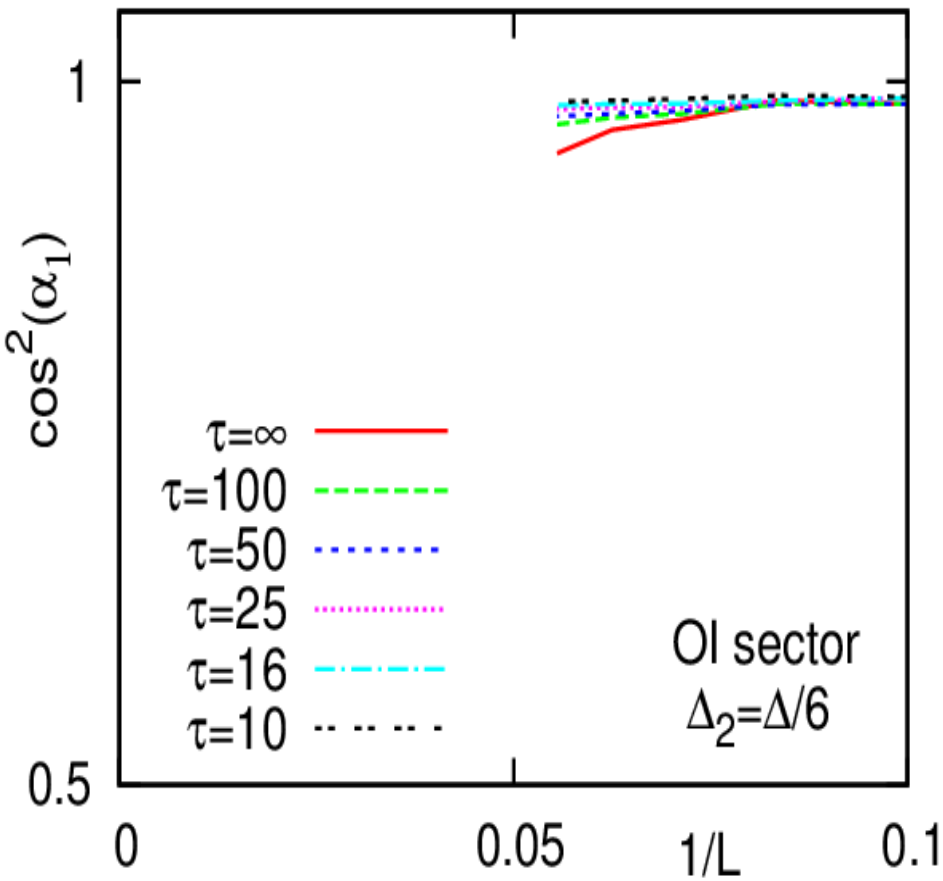
$$\Delta = 0.5, M = 5$$

$$\Delta_2 \neq 0 \quad \Delta_2 = 0$$

$$\langle \bar{A}^\tau A \rangle = \sum_l \lambda_l \boxed{\frac{\langle A Q_l^M \rangle^2}{(\langle Q_l^M)^2 \rangle}}$$

$$\cos(\alpha_l) = \frac{\langle Q_l^M Q_{l0}^M \rangle}{||Q_l^M||\ ||Q_{l0}^M||}$$



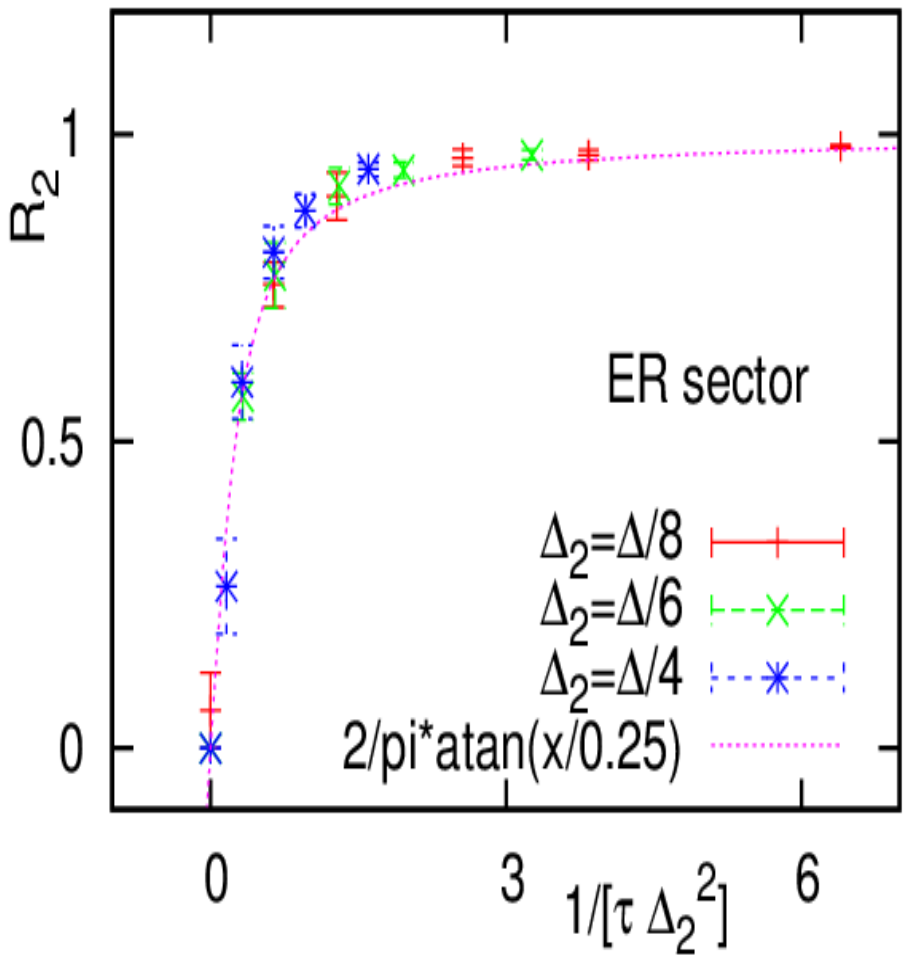


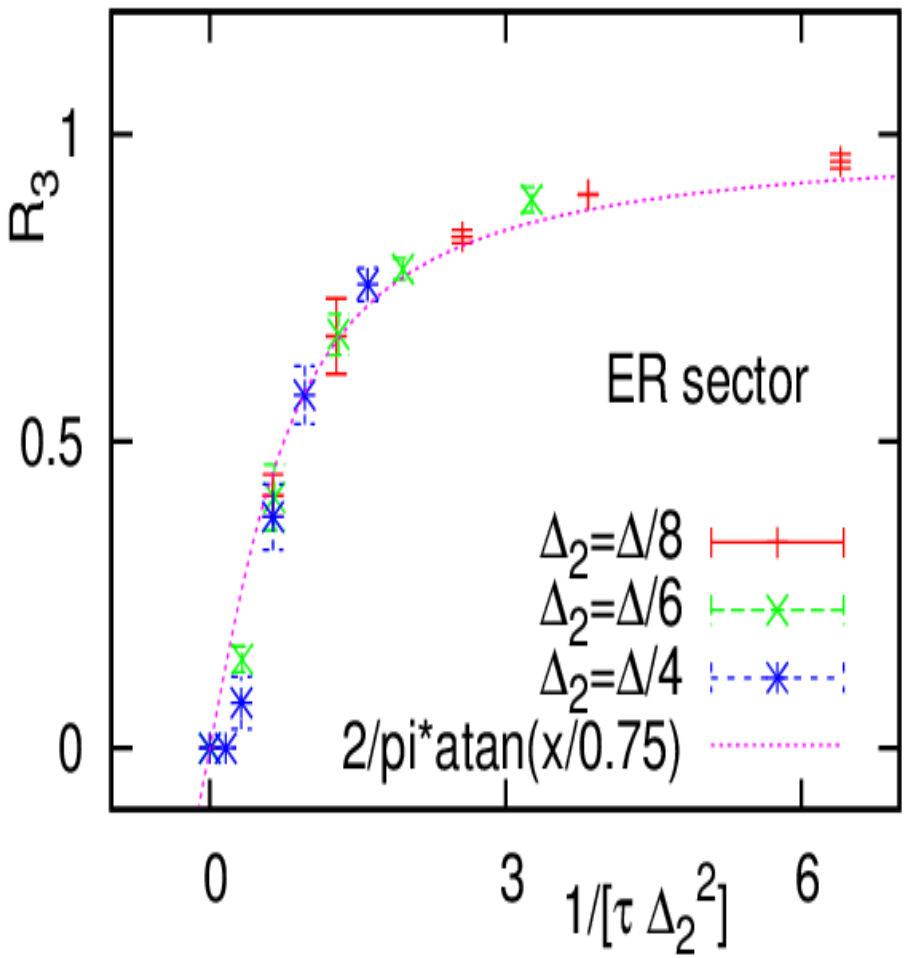
Dependence on Δ_2

$\Delta = 0.5, M = 5$

Lowest order perturbation should hold for weak Δ_2

$$R_l(\Delta_2, \tau) = \frac{\lambda_l(L \to \infty, \tau, \Delta_2)}{\lambda_l(L \to \infty, \tau \to \infty, 0\ \ )} \simeq \frac{2}{\pi} \arctan\left(\frac{1}{\Gamma_l\ \tau\ (\Delta_2)^2}\right)$$





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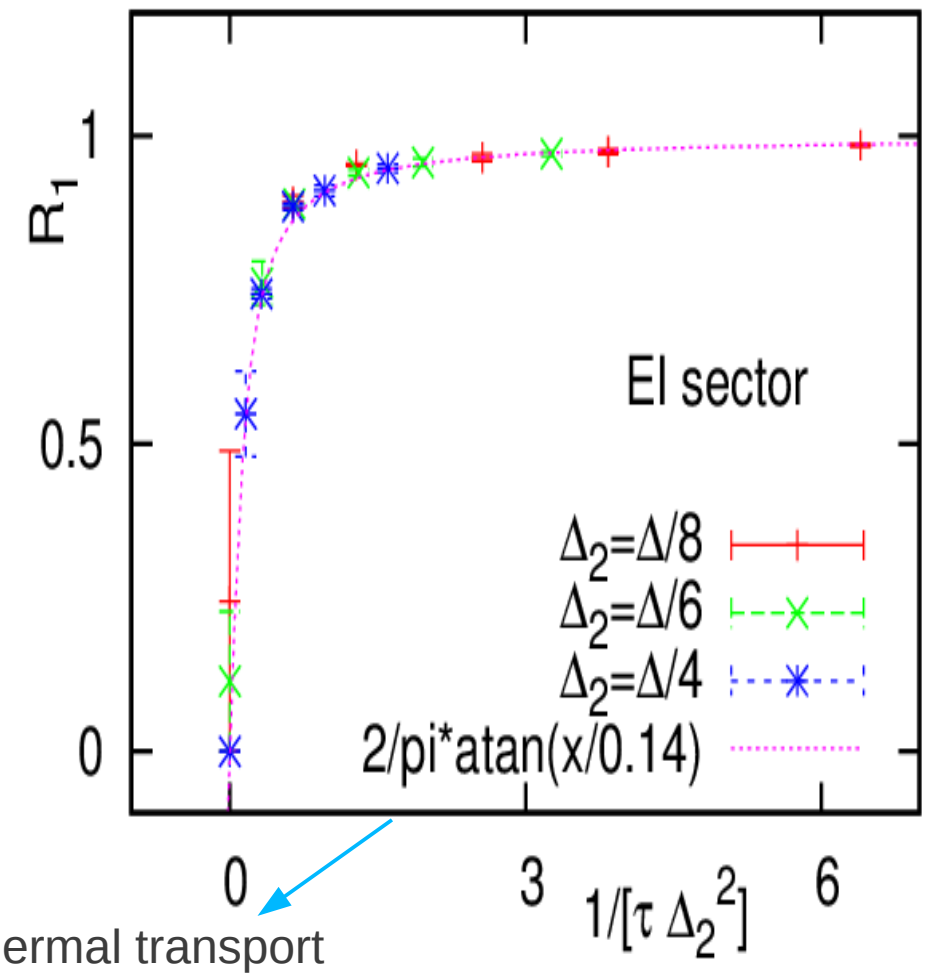
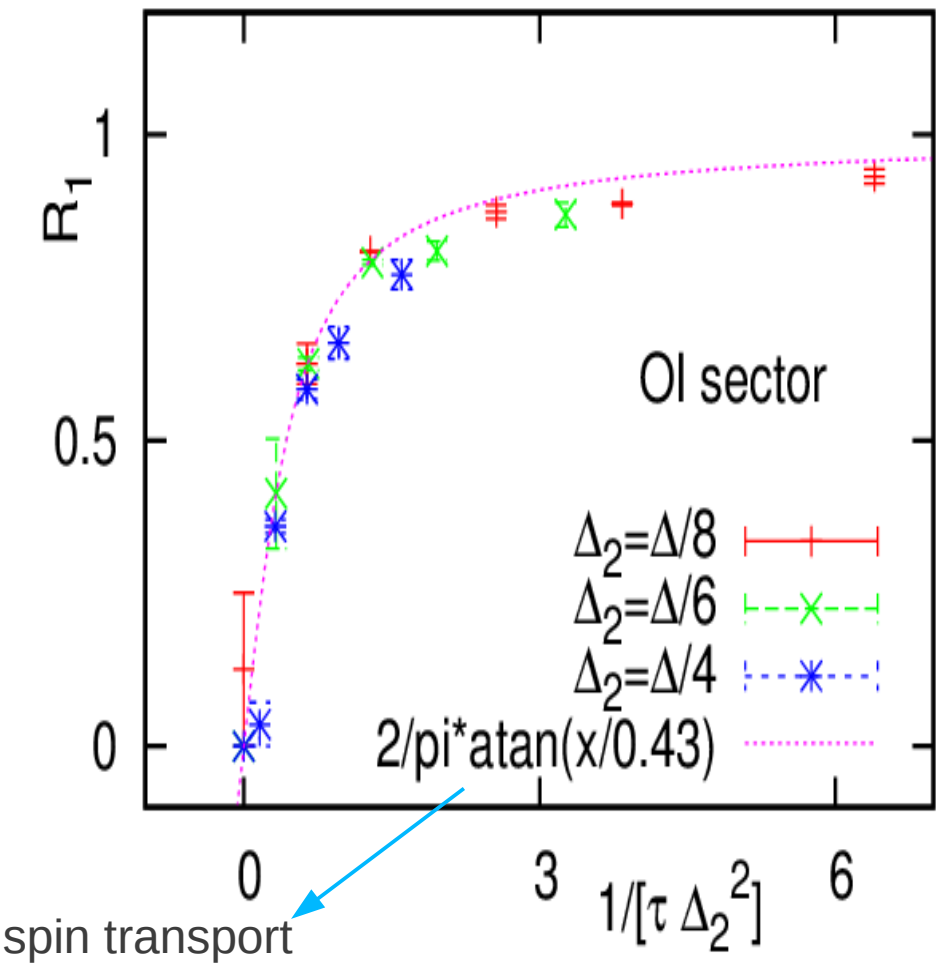


thermal transport

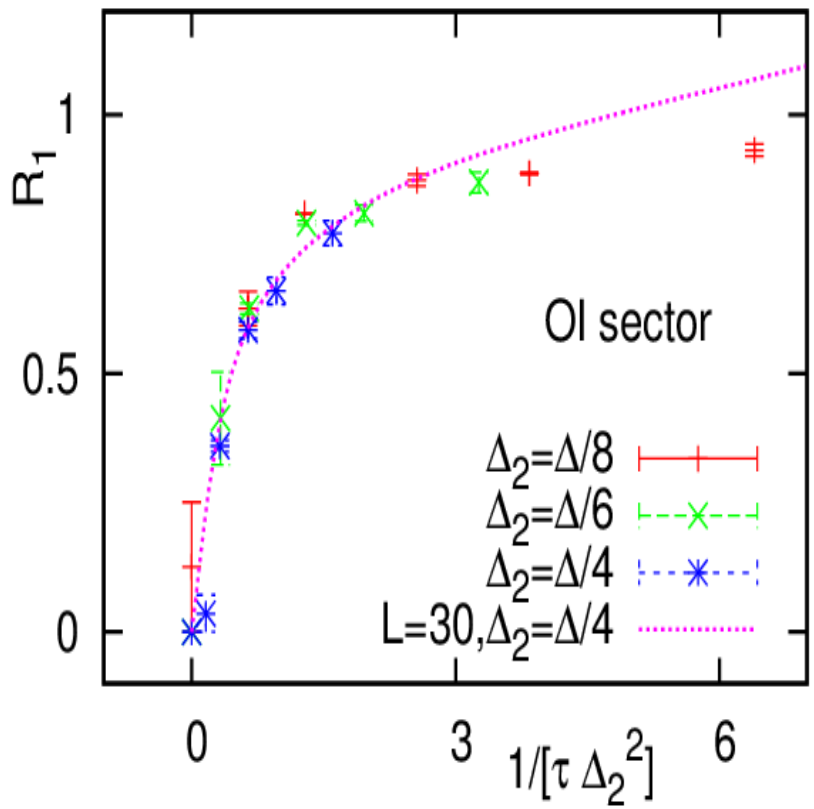


spin transport

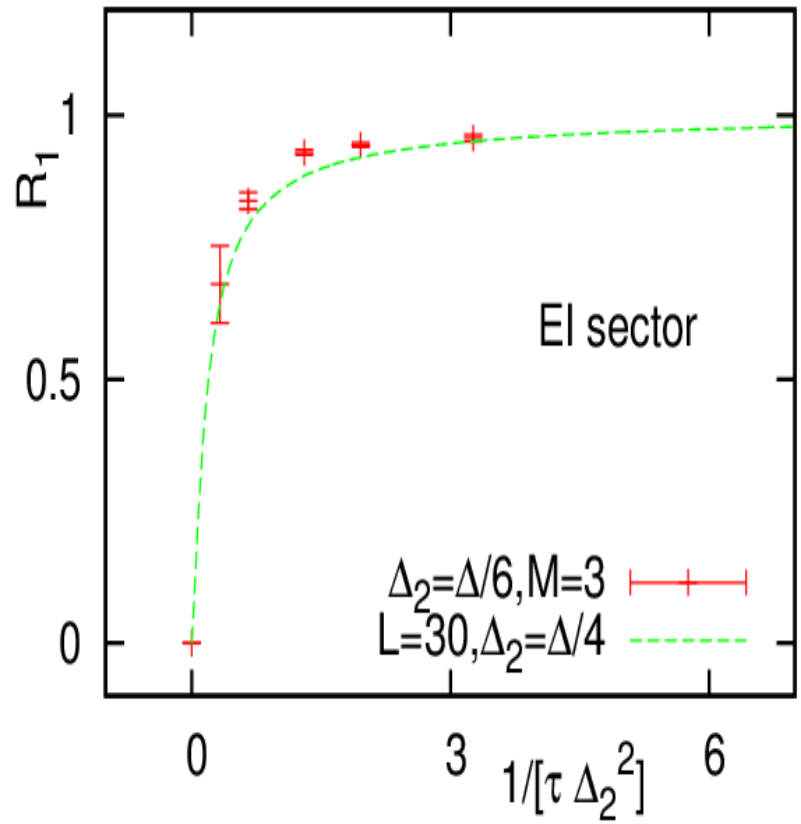
Test for L=30 (MCLM)

$$\lambda_1 \propto \langle \bar{J}^\tau J \rangle = \int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} d\omega \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle J(t) J \rangle$$

Rescaled data for spin current



Raw data for energy current



Main results for nearly integrable systems

$$\frac{1}{\tau}\int_0^\tau dt\langle A(t)B\rangle \sim \langle \bar{A}^\tau B\rangle = \sum_l \lambda_l \frac{\langle AQ_l^M\rangle^2}{\langle (Q_l^M)^2\rangle}$$

After breaking integrability by Δ_2

- smooth changing of $Q_l(\tau, \Delta_2)$
- all local Q_l become quasilocal (except for H)
- support of Q_l expands perturbatively as $\lambda_l =\propto \arctan\left(\frac{1}{\Gamma_l\ \tau\ (\Delta_2)^2}\right)$
- Γ_l increase with support, but are of the same order of magnitude (for small M)